Definite Integral

1. Definite Integral As The Limit Of A Sum

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+\dots+f(a+(n-1)h)]$$

$h = \frac{b-a}{n} \to 0$ as $n \to \infty$

The above expression is known as the definite integral as the limit of a sum.

2. Properties Of Definite Integrals

(i) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ in particular $\int_a^a f(x)dx = 0$

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a<c<b$

(iv) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(vi) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is an even function

(vii) $\int_{-a}^a f(x)dx = 0$, if $f(x)$ is an odd function

(viii) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

(ix) $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a-x) = f(x)$

$= 0$, if $f(2a-x) = -f(x)$

3. Walli's formula

- $$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\dots\dots\frac{2}{3}, & \text{when } n \text{ is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\dots\dots\dots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} & \text{when } n \text{ is even} \end{cases}$$

- $$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{(m-1)(m-3)\dots(n-1)(n-3)\dots}{(m+n)(m+n-2)\dots(2 \text{ or } 1)}$$

  [If m, n are both odd positive integers or one odd positive integer]

- $$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\dots\dots\dots\dots(n-1)(n-3)}{(m+n)(m+n-2)\dots\dots(2 \text{ or } 1)}\cdot\frac{\pi}{2}$$

  [If m, n are both even positive integers]

4. Periodic Properties

If f(x) is a periodic function with period T, then

01 $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx, n \in Z$

02 $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx, n \in Z, a \in R$

03 $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx, m,n \in Z$

04 $\int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)dx, n \in Z, a \in R$

5. Advance properties

$\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b \psi(x)dx \le \int_a^b f(x)dx \le \int_a^b \phi(x)$

01 If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

02 $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$

03 If $f(x) \ge 0$ on $[a,b]$ then $\int_a^b f(x)dx \ge 0$

6. Leibnitz Theorem

if $F(x) = \int_{g(x)}^{h(x)} f(t)dt$, then

$$\frac{dF(x)}{dx} = h'(x)f(h(x)) - g'(x)f(g(x))$$

Gamma function

$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$

where n is a positive rational number

7. Beta & Gama Function

Properties of gamma function

1) $\Gamma(0) = \infty, \Gamma(1) = 1$

2) $\Gamma(n+1) = n\Gamma(n)$

3) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4) $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}, 0<n<1$

Beta function

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

NOTE The relationship between beta & gamma function will be

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

8. Important results

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{n=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

In GP, sum of n terms, $S = s_n = \begin{cases} \frac{a(r^n-1)}{r-1}, & |r|>1 \\ an, & r=1 \\ \frac{a(1-r^n)}{1-r}, & |r|<1 \end{cases}$

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}, \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

$$\sin\alpha + \sin(\alpha+1\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+\overline{(n-1)}\beta) = \frac{\sin(n\beta/2)}{\sin(\beta/2)}\sin(\alpha+(n-1)\beta/2).$$

$$\cos\alpha + \cos(\alpha+\beta) + \cdots + \cos(\alpha+\overline{(n-1)}\beta) = \frac{\sin n\beta/2}{\sin\beta/2}\cdot\cos(\alpha+(n-1)\beta/2)$$

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\cdots\infty = \frac{\pi^2}{12}$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\infty = \frac{\pi^2}{6}$$

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\quad\infty = \frac{\pi^2}{24}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$1/1^2 + 1/3^2 + 1/5^2 + 1/7^2 + \ldots\infty = \pi^2/8$$

9 Average Value Theorem

If f is a continuous function on [a, b], then its average value on [a, b] is given by the formula

$$f_{\text{AVG}[a,b]} = \frac{1}{b-a}\cdot\int_a^b f(x)dx$$

Application Of Integrals

1

$\int_a^b f(x)dx \neq$ Area under the curve f(x) from a to b

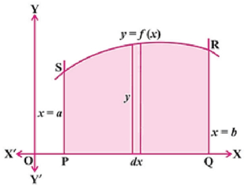
$\int_a^b f(x)dx =$ Algebraic area under the curve f(x) from a to b

2 POSITIVE AND NEGATIVE AREA

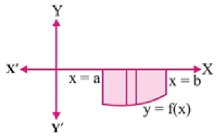
Area is always taken as positive. If some part of the area lies in the positive side i.e., above x-axis and some part lies in the negative side i.e. below x-axis then the area of two parts should be calculated separately and then add their numerical values to get the desired area.

3 Area Under Simple Curves

(i) Area of the region bounded by a curve y = f(x) and x-aixs between the two ordinates

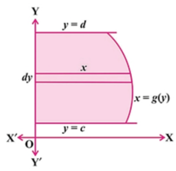


Area, $A = \int_a^b dA = \int_a^b y dx = \int_a^b f(x)dx$

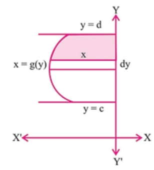


If the position of the curve under consideration is below the x -axis. Then, area is negative. So, we take its absolute value, i.e.,
$\text{Area}(A) = \left|\int_a^b f(x)dx\right|$

(ii) Area of the region bounded by a curve x = f(y) and x-aixs between the two ordinates



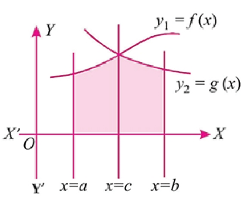
Area, $A = \int_c^d x dy = \int_c^d g(y)dy$



If the position of the curve under consideration is below the y-axis. Then, area is negative. So, we take its absolute value, i.e.,
$\text{Area}(A) = \left|\int_c^d g(y)dy\right|$

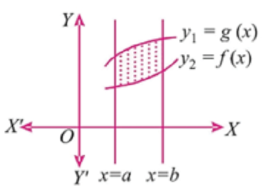
4 Area Under Different Curves

CASE-I



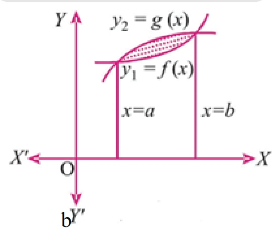
$$A = \int_a^c f(x)dx + \int_c^b g(x)dx$$

CASE-II



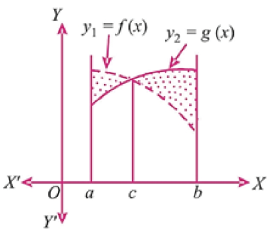
$$A = \int [g(x) - f(x)]dx$$

CASE-III



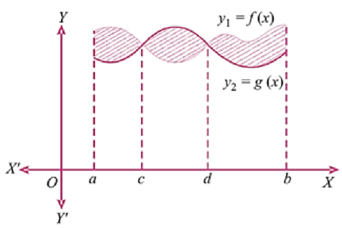
$$A = \int_a^b [g(x) - f(x)]dx$$

CASE-IV



$$A = \int_a^c \left[f(x) - g(x)\right]dx + \int_c^b \left[g(x) - f(x)\right]dx$$

CASE-V



$$A = \int_a^c (y_1 - y_2)dx + \int_c^d (y_2 - y_1)dx + \int_d^b (y_1 - y_2)dx$$